n Equation in n Variables

Liming Pang

1 Uniqueness of Solutions

As we have seen in some of the examples in the previous sections, a system of linear equations may not always have a unique solution. There may be no solution at all, or there may be more than one solutions. When the number of equations equals the number of variables, we can tell if the solution is unique by studying its corresponding matrix.

Theorem 1. *The system of linear equations*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ .... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases}$$

has a unique solution if and only if the matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

is invertible.

Example 2.

$$\begin{cases} x_1 + 2x_2 + 2x_3 = 1 \\ 3x_1 - x_2 + 2x_3 = 3 \\ -3x_2 - x_n = 5 \end{cases}$$

$$\det \begin{bmatrix} 1 & 2 & 2 \\ 3 & -1 & 2 \\ 0 & -3 & -1 \end{bmatrix} = -5$$

So the matrix is invertible, and there is a unique solution.

2 Cramer's Rule

There is another method to solve a system of linear equations when then corresponding matrix is invertible.

Theorem 3. *(Cramer's Rule) If the system of linear equations*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ .... \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n \end{cases}$$

has a unique solution, then the solution can be computed by

$$x_i = \frac{\det B_i}{detA}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

and B_i is obtained from A by replacing the i-th column by $\begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}$

Example 4. *We know the system of linear equations*

$$\begin{cases} ax + by = k_1 \\ cx + dy = k_2 \end{cases}$$

has unique solution if and only if

$$\det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \neq 0$$

So when $ad - bc \neq 0$, we can apply the Cramer's Rule to find the unique solution:

$$\det B_1 = \det \begin{bmatrix} k_1 & b \\ k_2 & d \end{bmatrix} = k_1 d - k_2 b, \det B_2 = \det \begin{bmatrix} a & k_1 \\ c & k_2 \end{bmatrix} = ak_2 - ck_1$$

So the solution is

$$x_1 = \frac{k_1 d - k_2 b}{ad - bc}, x_2 = \frac{ak_2 - ck_1}{ad - bc}$$

Remark 5. There is another way to obtain this unique solution for the example above:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

So

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} dk_1 - bk_2 \\ -ck_1 + ak_2 \end{bmatrix}$$

Example 6. *Solve the system of linear equations*

$$\begin{cases} 2x_2 - x_3 = -7 \\ x_1 + x_2 + 3x_3 = 2 \\ -3x_1 + 2x_2 + 2x_3 = -10 \end{cases}$$

First, we see

$$\det A = \det \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 3 \\ -3 & 2 & 2 \end{bmatrix} = -27$$

So the system of equations has a unique solution, we can apply the Cramer's Rule.

$$\det B_1 = \det \begin{bmatrix} -7 & 2 & -1 \\ 2 & 1 & 3 \\ -10 & 2 & 2 \end{bmatrix} = -54$$

$$\det B_2 = \det \begin{bmatrix} 0 & -7 & -1 \\ 1 & 2 & 3 \\ -3 & -10 & 2 \end{bmatrix} = 81$$

$$\det B_3 = \det \begin{bmatrix} 0 & 2 & -7 \\ 1 & 1 & 2 \\ -3 & 2 & -10 \end{bmatrix} = -27$$

So $x_1 = \frac{\det B_1}{\det A} = 2, x_2 = \frac{\det B_2}{\det A} = -3, x_3 = \frac{\det B_3}{\det A} = 1$

3 Nontrivial Solutions

Consider the system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0 \\ \text{..} \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = 0 \end{cases}$$

Obviously $x_1 = 0, x_2 = 0, ..., x_n = 0$ is a solution. We call it the trivial solution. Solutions other than this one are called nontrivial solution.

Theorem 7. *The system of linear equations*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = 0 \\ \text{..} \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = 0 \end{cases}$$

has nontrivial solutions if and only if $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$ *is not invertible, i.e.,* $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} = 0$

Example 8. *Find the value of λ for which the following system of equations has nontrivial solutions:*

$$\begin{cases} 5x + 2y + z = \lambda x \\ 2x + y = \lambda y \\ x + z = \lambda z \end{cases}$$

The system of equations can be written as

$$\begin{cases} (5 - \lambda)x + 2y + z = 0 \\ 2x + (1 - \lambda)y = 0 \\ x + (1 - \lambda)z = 0 \end{cases}$$

So it has nontrivial solution if and only if

$$\det \begin{bmatrix} (5-\lambda) & 2 & 1 \\ 2 & (1-\lambda) & 0 \\ 1 & 0 & (1-\lambda) \end{bmatrix} = \lambda(1-\lambda)(\lambda-6) = 0$$

So when $\lambda = 0, 1, 6$, there are nontrivial solution.

Remark 9. The above example will be important if you study the concept of eigenvalues and eigenvectors of a matrix in Linear Algebra course. The process above is exactly computing the eigenvalues of the matrix

$$\begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Theorem 10. *If* $\begin{bmatrix} x_1^0 \\ x_2^0 \\ \cdots \\ x_n^0 \end{bmatrix}$ *is a particular solution of the system of linear equations*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

Then the set of all solutions to the above system of linear equations are of the form

$$\begin{bmatrix} x_1^0 \\ x_2^0 \\ \cdots \\ x_n^0 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix}$$

where $\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix}$ *is any solution of the system of linear equations*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = 0 \end{cases}$$

4 The Leontief Model

One application of system of linear equations is the Leontief Model.

Assume there is an economy producing n types of goods. To produce each good, the corresponding industry needs at least one of the $n-1$ goods as input.

Let x_i denote the total amount of good i that is going to be produced. Let a_{ij} be the amount of good i needed in order to produce one unit of good j.

By the above assumption, we see in order to produce x_1 units of good 1,...,x_n units of good n, $a_{i1}x_1 + ... + a_{in}x_n$ units of good i is needed.

If we also require b_i units of good i to be supplied to the market besides those as input for other goods, we get the relation

$$x_i = a_{i1}x_1 + ... + a_{in}x_n + b_i$$

So $x_1, ..., x_n$ needs to satisfy the system of equations

$$\begin{cases} x_1 = a_{11}x_1 + ... + a_{1n}x_n + b_1 \\ x_2 = a_{21}x_1 + ... + a_{2n}x_n + b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ x_n = a_{n1}x_1 + ... + a_{nn}x_n + b_n \end{cases}$$

If we write

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}, \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{bmatrix}$$

the above equations can be written as

$$\vec{x} = A\vec{x} + \vec{b}$$

i.e.,

$$(I_n - A)\vec{x} = \vec{b}$$

We can then find $\vec{x}$ by solving this system of equations.